Math 005C Exam 2

1. (3 pts) Find the following limit

$$\lim_{(x,y)\to(1,1)} \frac{5x^2y}{\sqrt{x^2+4y^2}} = \frac{5(1)^2(1)}{\sqrt{1^2+4(1)^2}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

2. Consider the following limit.

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$$

a) (3 pts) Find the limit along $y = 0$.

$$\lim_{x\to 0} \frac{x^4}{x^4} = 1$$

b) (3 pts) Then find the limit along $y = x$.

$$\lim_{x\to 0} \frac{(x^2-x^2)^2}{x^2+x^2} = 0$$

c) (2 pts) What can you conclude about the limit?

The limit does not exist.

3. (2 pts each) Find f_x, f_y, f_{xx}, and f_{xy}.
$f(x,y) = x^4 - 4x^2y^3$

$$f_x = 4x^3 - 8xy^3$$

$$f_y = -12x^2y^2$$

$$f_{xx} = 12x^2 - 8y^3$$

$$f_{xy} = -24xy^2$$

4. (10 pts) Find the directional derivative of $f(x,y) = 2\sqrt{x} - y^3$ in the direction of $\langle 2, -2 \rangle$ at the point $(1,3)$.

$$\vec{V} = \langle 2, -2 \rangle \Rightarrow \vec{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$D_{\vec{u}} f(x,y) = \left\langle \frac{1}{\sqrt{x}}, -3y^2 \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$D_{\vec{u}} f(1,3) = \langle 1, -27 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{1}{\sqrt{2}} + \frac{27}{\sqrt{2}} = \frac{28}{\sqrt{2}}$$

$$\text{OR} \quad = 14\sqrt{2}$$

5. (6 pts each) Suppose the pressure in Pascals at point (x, y, z) in a fluid is given by $P(x, y, z) = 100\, e^{-3x^2-y^2-2z^2}$ where x, y, and z are in meters. At the point $Q(1, 1, 0)$:

a. Find the direction in which the pressure increases most rapidly.

$$\vec{\nabla} f = \langle -600x, -200y, -400z \rangle e^{-3x^2-y^2-2z^2}$$

$$\vec{\nabla} f(1,1,0) = \langle -600e^{-4}, -200e^{-4}, 0 \rangle$$

The direction is $\langle -3, -1, 0 \rangle$.

b. Find the maximum rate of change at Q.

$$\|\vec{\nabla} f(1,1,0)\| = 200e^{-4}\sqrt{3^2+1^2}$$

$$= 200\sqrt{10}\, e^{-4}$$

c. Moving from the point $(2, 3, 4)$ in the positive y direction, is the pressure increasing or decreasing? Be sure to justify your answer.

$$\vec{\nabla} f(2,3,4) \cdot \langle 0,1,0 \rangle$$

$$= -600\, e^{-53} \langle 0, 1, 0 \rangle$$

$$= -600 e^{-53} \text{ (negative)}$$

So it is decreasing.

6. (12 pts) Given $z = x^3 + 2xy^2$. If (x, y) changes from $(2, 1)$ to $(2.05, 0.9)$, find and compare the values of Δz and dz.

$$\Delta z = f(2.05, 0.9) - f(2, 1)$$

$$= 2.05^3 + 2(2.05)(0.9)^2 - \left[2^3 + 2(2)(1)^2\right]$$

$$\approx \underline{-0.063875}$$

$$dz = f_x\,dx + f_y\,dy = (3x^2 + 2y^2)dx + 4xy\,dy$$

$$(x, y) = (2, 1), \quad dx = 0.05, \quad dy = -0.1$$

$$\Rightarrow dz = (3 \cdot 2^2 + 2 \cdot 1^2)(0.05) + 4(2)(1)(-0.1) = \underline{-0.1}$$

Since $-0.1 < -0.0639$,

$dz < \Delta z$.

7. (10 pts) Use a linear approximation or differentials to estimate the value of $\sqrt{8.9} + \sqrt[3]{26.8}$ to 6 decimal places.

Linear approximation:

$$f(x, y) = \sqrt{x} + \sqrt[3]{y} = x^{1/2} + y^{1/3}$$

$$f_x = \frac{1}{2\sqrt{x}}, \quad f_y = \frac{1}{3y^{2/3}}$$

$$L(x, y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Use $a = 9$, $b = 27$, $x = 8.9$, $y = 26.8$

$$L(8.9, 26.8) = 3 + 3 + \frac{1}{6}(-0.1) + \frac{1}{27}(-0.2)$$

$$\approx 5.975926$$

For reference, $\sqrt{8.9} + \sqrt[3]{26.8} \approx 5.975861$

Differentials: $dz = f_x dx + f_y dy$

$$= \frac{1}{2\sqrt{9}}(-0.1) + \frac{1}{27}(-0.2)$$

$z \approx f(9, 81) + dz$ (same result)

8. (10 pts) Find the critical points of the function $f(x, y) = x^2 - xy + y^2 - 2x$. Classify each point as a local maximum, a local minimum, or a saddle point. You do not need to find the function value(s) at the critical point(s).

$f_x = 2x - y - 2, \quad f_y = -x + 2y$

$f_x = 0 \Rightarrow 2x - y - 2 = 0 \quad (1)$

$f_y = 0 \Rightarrow -x + 2y = 0 \quad (2)$

Eq. 1 + 2 Eq 2:

$$\begin{array}{r} 2x - y = 2 \\ -2x + 4y = 0 \\ \hline 3y = 2 \Rightarrow y = \frac{2}{3} \end{array}$$

then Eq 1: $2x = \frac{8}{3} \Rightarrow x = \frac{4}{3}$

$\left(\frac{4}{3}, \frac{2}{3}\right)$ is a critical point.

$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -1$

$D(x,y) = 2(2) - (-1)^2 > 0$

with $f_{xx} = 2 > 0$, so

there is a local minimum at $\left(\frac{4}{3}, \frac{2}{3}\right)$.

9. (8 pts) Use the method of Lagrange multipliers to find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $x + 2y = 5$.

$$\vec{\nabla} f = \langle 2x, 2y \rangle, \quad \vec{\nabla} g = \langle 1, 2 \rangle$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$2x = \lambda \qquad (1)$$

$$2y = 2\lambda \qquad (2)$$

$$x + 2y = 5 \qquad (3)$$

Eq 1 + Eq 2: $2x = y$

$$x + 4x = 5 \Rightarrow x = 1$$

$$2x = y \Rightarrow y = 2$$

$$f(1, 2) = 1^2 + 2^2 = 5$$

The minimum value is 5.